PHYSICAL REVIEW E **80**, 036603 (2009)

Breather-breather interactions in sine-Gordon systems using collective coordinate approach

Munehiro Nishida,[1] Yoshiki Furukawa,[1] Toshiyuki Fujii,[2] and Noriyuki Hatakenaka[2]
[1]*Graduate School of Advanced Sciences of Matter, Hiroshima University, Higashi-Hiroshima 739-8530, Japan*
[2]*Graduate School of Integrated Arts and Sciences, Hiroshima University, Higashi-Hiroshima 739-8521, Japan*
(Received 12 May 2009; published 14 September 2009)

We investigate the interaction between breathers in sine-Gordon systems using a collective coordinate approach. Focusing on the in-phase or out-of-phase oscillation mode of two identical breathers, we derive a simple ordinary differential equation for the collective coordinate: the separation between the centers of mass of the two breathers R. Analytic solutions of this equation can reproduce quantitatively the results of a direct numerical simulation of the sine-Gordon equation over the whole parameter range. The interaction between the two breathers is attractive (repulsive) for the in-phase (out-of-phase) oscillation mode with an asymptotic exponential dependence on R. We find that the interaction within the innermost kink-antikink (kink-kink) pair for the in-phase (out-of-phase) case plays the most significant role in determining the sign and the strength of the effective breather-breather interaction. We also find an internal oscillation mode of the in-phase oscillating breather pair and obtain an analytic expression for the angular frequency of the mode.

DOI: 10.1103/PhysRevE.80.036603 PACS number(s): 05.45.Yv, 63.20.Pw

I. INTRODUCTION

Elementary excitations play a crucial role with respect to the nature of condensed matters. In integrable nonlinear systems, elementary excitations are nonlinear localized modes, such as solitons and solitary waves. One such excitation, a breather in a sine-Gordon (SG) system, is an oscillating solitary wave, namely, a spatially localized oscillating mode, which consists of a bound pair of topological solitons, a kink and an antikink. Recently, a lot of attention has focused on the roles played by the breathers in real materials, which are well described by the SG equation. It has been found that the breathers do exist and directly affect the electronic, optical, and transport properties of various materials, such as Josephson superconducting junctions [1,2], charge density wave systems [3], 4-methyl-pyridine crystals [4], conjugated polymers [5], and antiferromagnetic Heisenberg chains [6,7].

Although no experimental evidence has been reported, "many-breather effects," such as the appearance of new collective modes in coupled multibreather systems, can be expected when breathers become dense. Exact analytic solutions for multiple breathers in the SG equation are obtained by using a Bäcklund transformation since the SG equation is an integrable equation. However, direct knowledge of the breather-breather interaction would be more helpful in terms of understanding the physical phenomena related to many-breather effects in real materials since the presence of many perturbations prevents us from applying the exact solutions directly.

Direct numerical simulations of the SG equation can reproduce the dynamical behavior of multibreather systems under arbitrary perturbations. However, it is difficult to extract the essential properties of multibreather dynamics from the numerical results. Thus, some methods are needed to construct a simple model that greatly reduces the information involved in the system and provide core information for understanding the mechanisms of the breather dynamics. One such method is the collective coordinate (CC) method, which employs a set of parameters as CCs, such as center-of-mass position or total momentum [8–12]. We can achieve a great reduction from the originally infinite-dimensional problem to a low dimensional system of ordinary differential equations (ODEs) for the CCs. The CC approach enables us to obtain analytic expressions for multibreather properties, which would be useful for the interpretation of experimental data and for the prediction of material properties.

The CC approach is especially effective for the quantization of breathers since it maps the physics of a quantum field onto the quantum mechanics of a single particle described by the CCs. It has been proposed that a quantized breather in a long Josephson junction can be utilized as a massive mobile qubit that acts as a quantum data bus and transfers quantum information from one node to another [13]. The breather-breather interaction plays a crucial role in the coherent manipulation of quantum information by creating an entanglement between breathers. In this paper, we investigate the interaction between breathers in SG systems using CC approaches.

To date, there have been only a few reports concerning the breather-breather interaction in an SG system. Pioneering work was undertaken by Kevrekidis *et al.* They studied the breather-breather interaction using both the exact breather-lattice solution [14] and Manton's approach [15,16]. They focused on *in-phase* oscillation where the SG phase field for the breathers is always symmetric around the center of the pair, as shown in Fig. 1(a), and showed that the breather-breather interaction is attractive and exhibits asymptotic exponential dependence on the breather separation and power-law dependence on the breather frequency. However, their approaches have certain limitations. The breather-lattice approach [14] is inapplicable to the problem of the interaction between out-of-phase oscillating breathers since there are no known out-of-phase breather-lattice solutions. In addition, Manton's approach [15] cannot be directly applied in the presence of spatially dependent potentials.

Our CC approach can deal equally well with the problems of in-phase and out-of-phase breather pairs. This approach is applicable to systems with spatially dependent potentials, al-

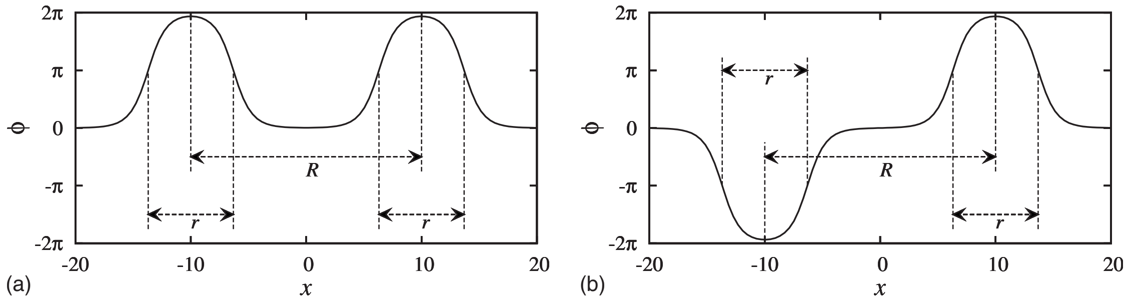

FIG. 1. Profile of phase fields for breather pairs of in-phase oscillation (a) and out-of-phase oscillation (b). The phase field for the in-phase (out-of-phase) pair is always symmetric (antisymmetric) around the center of the pair.

though we deal only with homogeneous systems here. The SG equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin \phi = 0 \tag{1}$$

has two kinds of soliton solutions, i.e., kink and antikink solutions, which are expressed as

$$\sigma(x) = 4 \arctan[\exp(x)], \quad \text{kink}, \tag{2}$$

$$\bar{\sigma}(x) = 4 \arctan[\exp(-x)], \quad \text{antikink}. \tag{3}$$

Here, all quantities are properly nondimensionalized. It has been shown that the kink soliton has excellent stability against various perturbations and its dynamics are essentially those of a classical particle [17]. There is another exact solution of Eq. (1), i.e., a breather solution, which describes the internal oscillation of the bound kink-antikink pair and is expressed as

$$\phi_b(x,t) = 4 \arctan\left[\frac{\Gamma \cos(\omega t)}{\omega \cosh(\Gamma x)}\right], \tag{4}$$

where ω denotes the frequency of the breather internal oscillation and Γ is determined from ω by the relation $\omega^2+\Gamma^2=1$. The breather solution [Eq. (4)] can be constructed by the superposition of kink and antikink solutions given in Eqs. (2) and (3), namely,

$$\begin{aligned}\phi_{\text{kp}}(x,t) &= \sigma\{\Gamma[x - r(t)/2]\} + \bar{\sigma}\{\Gamma[x + r(t)/2]\} - 2\pi \\ &= \sigma\{\Gamma[x - r(t)/2]\} - \sigma\{\Gamma[x + r(t)/2]\},\end{aligned} \tag{5}$$

where Γ is the parameter that determines the typical width of a kink by $1/\Gamma$ and r denotes the separation between the centers of mass of the kink and the antikink. It has been shown that Eq. (5) is coincident with Eq. (4) if r is assumed to obey Matsuda's identity [9,10],

$$r(t) = \frac{2}{\Gamma}\text{arcsinh}\left[\frac{\Gamma}{\omega}\cos(\omega t)\right]. \tag{6}$$

Thus, the breather can be regarded as a kind of composite particle composed of a kink and an antikink. Therefore, it is natural to use the separation r as the CC that describes the breather internal oscillation. Indeed, this CC has been used in several studies of the dynamics of a single breather under the influence of static and dynamic perturbations [9–12]. Based on this approach, we express a breather pair as a pair of two kink-antikink pairs and introduce a set of CCs: the width of the individual breathers, r, and the separation between the centers of mass of the two breathers, R, as shown in Fig. 1. Assuming that the SG phase field ϕ is expressed by the superposition of two kink-antikink-pair solutions in which the width of each pair is r and the separation between the pairs is R, we obtain a set of ODEs that describe the dynamics of these CCs.

Caputo and Flytzanis [18] pointed out that some two collective coordinate *Ansätze* introduce a mathematical singularity. This singularity arises when the breather becomes flat. Unfortunately, this also applies to our *Ansatz*, and the inertial mass for R becomes zero when $r=0$, namely, the breathers become flat. Fortunately, we can remove this singularity by using an appropriate coordinate transformation around the singularity [19] and can integrate the ODEs for the CCs accurately enough to reproduce the results of a direct numerical simulation of the SG equation.

The full set of our ODEs for the CCs contains overdetailed information, which is unnecessary for elucidating the effective interaction between the breathers. We derive a coarse-grained equation of motion by averaging in time assuming that the widths of the individual breathers are not affected by the breather-breather interaction. We obtain an analytic expression of the effective breather-breather interaction and show that the out-of-phase (in-phase) breather pair has a repulsive (attractive) interaction that exhibits an asymptotic exponential dependence on the breather separation.

The paper is organized as follows. In Sec. II, the collective coordinate approach for a single breather is briefly reviewed. In Sec. III, we introduce a set of CCs for a breather pair and derive a set of ODEs that describes the dynamics of these CCs. In Sec. IV, we discuss how to remove the singular points of our ODEs. In Sec. V, we derive a coarse-grained equation of motion. We obtain analytic solutions for the time dependence of R with various parameter values and compare them with the results of direct numerical simulations of the SG equation. Finally, Sec. VI discusses the results and provides a summary.

II. COLLECTIVE COORDINATES FOR SINGLE BREATHER

In order to derive the Lagrangian for the CC r, we substitute Eq. (5) into the Lagrangian of the SG phase field, which is expressed as

$$L[\phi] = K[\phi] - V[\phi], \tag{7}$$

$$K[\phi] \equiv \int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 dx, \tag{8}$$

$$V[\phi] \equiv \int_{-\infty}^{\infty} \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + (1 - \cos\phi)\right] dx. \tag{9}$$

The integration can be carried out by using the formulas [12]

$$f(y) \equiv \int_{-\infty}^{\infty} \frac{1}{\cosh(2\xi) + \cosh y} d\xi = \frac{y}{\sinh y}, \tag{10}$$

$$g(y) \equiv \int_{-\infty}^{\infty} \frac{1}{\{\cosh(2\xi) + \cosh y\}^2} d\xi = -\frac{f'(y)}{\sinh y} = \frac{y \coth y - 1}{\sinh^2 y}, \tag{11}$$

and the Lagrangian for r is derived as

$$L_{\mathrm{kp}}(r,\dot{r}) \equiv L[\phi_{\mathrm{kp}}] = K_{\mathrm{kp}}(r,\dot{r}) - V_{\mathrm{kp}}(r), \tag{12}$$

$$K_{\mathrm{kp}}(r,\dot{r}) \equiv K[\phi_{\mathrm{kp}}] = \tfrac{1}{2} M_{\mathrm{kp}}(r)\dot{r}^2, \tag{13}$$

$$M_{\mathrm{kp}}(r) \equiv 4\Gamma[1 + f(\Gamma r)], \tag{14}$$

$$V_{\mathrm{kp}}(r) \equiv V[\phi_{\mathrm{kp}}] = 8\Gamma\{1 - f(\Gamma r)\} + \frac{8}{\Gamma}[1 - f(\Gamma r) + 2\{\cosh(\Gamma r) - 1\} g(\Gamma r)], \tag{15}$$

where the dot denotes the differentiation with respect to t. The equation of motion for r is derived from the Euler-Lagrange equation,

$$\frac{d}{dt}\frac{\partial L_{\mathrm{kp}}}{\partial \dot{r}} - \frac{\partial L_{\mathrm{kp}}}{\partial r} = 0. \tag{16}$$

It should be noted that Matsuda's identity (6) obeys this equation. Therefore, the solution of Eq. (16) with the assumed form [Eq. (5)] indeed constructs the exact breather solution [Eq. (4)]. This formalism has been applied successfully to studies of the effect of external fields on breather dynamics, e.g., the dissociation of a breather into a kink and an antikink [10] or the behavior of temporal chaos [11] under an alternating-current driver.

The r dependence of the inertial mass for the relative motion between the kink and the antikink [Eq. (14)] makes it difficult to analyze Eq. (16). However, we see that the relevant part of the mass is simply a reduced mass 4Γ derived from the well-known mass of a single kink, 8Γ. The Lagrangian L_{kp} is expressed as the sum of two elements,

$$L_{\mathrm{kp}} = L_{\mathrm{kp}}^{(1)} + L_{\mathrm{kp}}^{(2)}, \tag{17}$$

$$L_{\mathrm{kp}}^{(1)} = 2\Gamma\dot{r}^2 - 8(\Gamma + \Gamma^{-1}) + \frac{16}{\Gamma}\frac{1}{1 + \cosh(\Gamma r)}, \tag{18}$$

$$L_{\mathrm{kp}}^{(2)} = f(\Gamma r)\{L_{\mathrm{kp}}^{(1)} + 16\Gamma\}. \tag{19}$$

The Euler-Lagrange equation reads

$$[1 + f(\Gamma r)]\left[\frac{d}{dt}\frac{\partial L_{\mathrm{kp}}^{(1)}}{\partial \dot{r}} - \frac{\partial L_{\mathrm{kp}}^{(1)}}{\partial r}\right] = \frac{\partial f(\Gamma r)}{\partial r}[16\Gamma - H_{\mathrm{kp}}^{(1)}], \tag{20}$$

$$H_{\mathrm{kp}}^{(1)} = 2\Gamma\dot{r}^2 + 8(\Gamma + \Gamma^{-1}) - \frac{16}{\Gamma}\frac{1}{1 + \cosh(\Gamma r)}, \tag{21}$$

where $H_{\mathrm{kp}}^{(1)}$ is the Hamiltonian corresponding to $L_{\mathrm{kp}}^{(1)}$. Since $H_{\mathrm{kp}}^{(1)}$ has a value of 16Γ with the exact breather solution, the right-hand side of Eq. (20) becomes zero, and the equation of motion reduces to

$$\frac{d}{dt}\frac{\partial L_{\mathrm{kp}}^{(1)}}{\partial \dot{r}} - \frac{\partial L_{\mathrm{kp}}^{(1)}}{\partial r} = 0. \tag{22}$$

This is simply the Euler-Lagrange equation for $L_{\mathrm{kp}}^{(1)}$, namely, the equation of motion for a particle of mass 4Γ in the potential,

$$V_{\mathrm{kp}}^{(1)}(r) = -\frac{16}{\Gamma}\frac{1}{1 + \cosh(\Gamma r)}. \tag{23}$$

Thus, the motion of r is considered to be the motion of a particle of mass 4Γ.

III. COLLECTIVE COORDINATES FOR A BREATHER PAIR

A similar procedure is used to introduce CCs for the breather pair. Figure 1 shows the profile of the SG phase field ϕ for the breather pair with in-phase (a) or out-of-phase (b) oscillation. Since each breather consists of a kink-antikink pair, we construct the phase of a breather pair with four kink solutions as

$$\phi_{\mathrm{bp}}^{\pm}(x,t) = \sigma\left[\Gamma\left(x + \frac{R+r}{2}\right)\right] - \sigma\left[\Gamma\left(x + \frac{R-r}{2}\right)\right] \pm \sigma\left[\Gamma\left(x - \frac{R-r}{2}\right)\right] \mp \sigma\left[\Gamma\left(x - \frac{R+r}{2}\right)\right], \tag{24}$$

where the upper (lower) sign corresponds to the in-phase (out-of-phase) oscillation mode. The CCs r and R denote the width of each breather and the separation between the breathers, respectively, as shown in Fig. 1. As with a kink-antikink pair, we can derive the Lagrangian for a breather pair by substituting Eq. (24) into the SG equation [Eq. (1)]. After some calculations, we have derived an analytic expression of the Lagrangian for r and R. The full results are summarized in Appendix A. Here, we restrict ourselves to the limit of a large separation, i.e., $\exp(-\Gamma|R|) \ll 1$, and show an asymptotic form,

$$L_{\mathrm{bp}}^{\pm} = L[\phi_{\mathrm{bp}}^{\pm}] = \tfrac{1}{2}M_R^{\pm}\dot{R}^2 + \tfrac{1}{2}M_r^{\pm}\dot{r}^2 + M_{Rr}^{\pm}\dot{R}\dot{r} - V_{\mathrm{bp}}^{\pm}, \tag{25}$$

$$M_R^{\pm} \simeq 8\Gamma[1 - f(\Gamma r)] \pm 16\Gamma e^{-\Gamma|R|} \times \{\Gamma R \cosh(\Gamma r) - \Gamma r \sinh(\Gamma r) - \Gamma R\}, \quad (26)$$

$$M_r^{\pm} \simeq 2M_{\mathrm{kp}}(r) \pm 16\Gamma e^{-\Gamma|R|}\{\Gamma R \cosh(\Gamma r) - \Gamma r \sinh(\Gamma r) + \Gamma R\}, \quad (27)$$

$$M_{Rr}^{\pm} \simeq \pm 16\Gamma e^{-\Gamma|R|}\{\Gamma r \cosh(\Gamma r) - \Gamma R \sinh(\Gamma r)\}, \quad (28)$$

$$V_{\mathrm{bp}}^{\pm} \simeq 2V_{\mathrm{kp}}(r) \pm \frac{32}{\Gamma} e^{-\Gamma|R|}\left[2\left\{3 - \cosh(\Gamma r) - \frac{4}{1+\cosh(\Gamma r)} + \Gamma r \frac{\sinh\frac{\Gamma r}{2}}{\cosh^3\frac{\Gamma r}{2}}\right\} + \omega^2\{\Gamma R \cosh(\Gamma r) - \Gamma r \sinh(\Gamma r) - \Gamma R\}\right], \quad (29)$$

where $M_R^{\pm}$, $M_r^{\pm}$, and $M_{Rr}^{\pm}$ are the inertial masses and $V_{\mathrm{bp}}^{\pm}$ is the potential energy for the collective coordinates, R and r.

IV. SINGULARITY

The inertial mass $M_R^{\pm}$ becomes zero when $r=0$, namely, the breathers become flat. This causes a blowup in the separation between breathers and introduces fatal errors into the numerical integration processes. This singularity comes from the fact that the separation between breathers is ill-defined when the breathers are flat.

Caputo and Flytzanis [18] pointed out that this type of singularity appears if the equations of motion for the CCs are obtained through a projection on a vector that becomes zero at one point in the evolution. However, Caputo *et al.* showed that the singularity for the SG breather and the kink-antikink solutions in ϕ^4 systems can be removed by proper coordinate transformations [19]. We show that the singularity in our problem can be removed by taking a similar approach to that described in [19].

Assuming $\Gamma r \ll 1$, we expand the masses [Eqs. (26)–(28)] and the potential [Eq. (29)] with respect to r around $r=0$ up to the second order. Omitting the terms containing $e^{-\Gamma|R|}$, we obtain the approximated Lagrangian $\tilde{L}_{\mathrm{bp}}$, which is expressed as

$$\tilde{L}_{\mathrm{bp}} = \tfrac{2}{3}\Gamma^3 r^2 \dot{R}^2 + 8\Gamma \dot{r}^2 - \tfrac{8}{3}(3+\Gamma^2)\Gamma r^2. \quad (30)$$

This can be rewritten by the coordinate $\theta=\Gamma R/\sqrt{12}$ as

$$\tilde{L}_{\mathrm{bp}} = \frac{16\Gamma}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{16\Gamma}{2}\left(1+\frac{\Gamma^2}{3}\right)r^2. \quad (31)$$

This is a Lagrangian for a two-dimensional harmonic oscillator in polar coordinates (r,θ) with an inertial mass 16Γ and a restoring force coefficient $16\Gamma(1+\Gamma^2/3)$. If we use Cartesian coordinates $(x,y)=(r\cos\theta, r\sin\theta) = [r\cos(\Gamma R/\sqrt{12}), r\sin(\Gamma R/\sqrt{12})]$, we obtain

$$\tilde{L}_{\mathrm{bp}} = \frac{16\Gamma}{2}(\dot{x}^2 + \dot{y}^2) - \frac{16\Gamma}{2}\left(1+\frac{\Gamma^2}{3}\right)(x^2+y^2), \quad (32)$$

which has no singularity. By using Eq. (32) around the singular points, we can remove the singularity and continue the numerical integration.

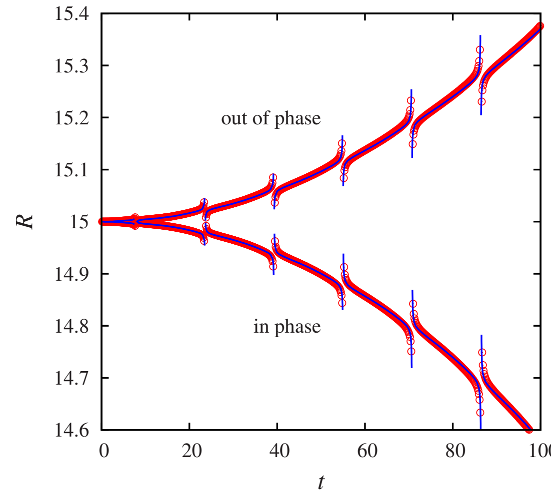

FIG. 2. (Color online) Time evolution of the separation between two breathers. The solid blue lines show the result of the numerical simulation for the sine-Gordon equation. The red circles show the results obtained by the numerical integration of the Euler-Lagrange equation for the collective coordinates removing singular points. Here, only data where the phase field amplitude is larger than 0.3 are selected.

Figure 2 shows the time evolutions of R obtained by the numerical integration of the Euler-Lagrange equation for L_{bp} removing singular points (red circles) and by direct numerical simulations of SG equation (blue lines). Here, to avoid a complicated figure, we only select data where the phase field amplitude is larger than 0.3. We adopt the Fourier pseudospectral method with a symplectic integrator for the SG simulation [20]. The time evolution of 32 768 Fourier modes is calculated with the velocity Verlet integration scheme. The system size and the time step used in the simulation are 256 and 1/256, respectively. The initial condition is obtained from Eq. (24) with some arbitrary values of ω and R (in Fig. 2, $\omega=0.2$ and $R=15$) and with r being set to the maximum of Matsuda's identity [Eq. (6)], namely, $(2/\Gamma)\mathrm{arcsinh}(\Gamma/\omega)$. In the simulations, we estimated the positions of the breathers from the peak positions of the phase profile by numerical searching for the absolute value maxima, one each from the left and right sides of the center of the breather pair, and then calculated the separation R from these positions. From Fig. 2 we find that the two results are in good agreement and indicate that the interaction between breathers oscillating out of phase (in phase) with each other is repulsive (attractive).

The discontinuous points in Fig. 2 correspond to the singular points mentioned above. This singularity is not an artifact introduced by the collective coordinate method since

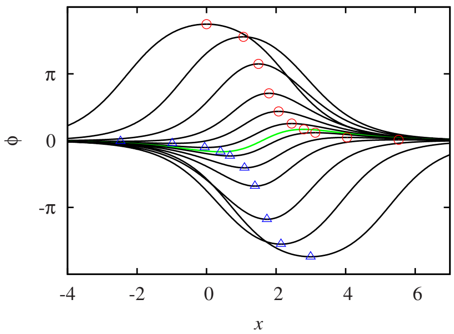

FIG. 3. (Color online) Profiles of the phase field of a single moving breather at different times. Time evolves from top curve to bottom curve. The red circles (blue triangles) that correspond to the peak (dip) positions are shown as a guide for the eyes.

the same behavior also appears in the direct numerical simulation of the SG equation. We can understand this singular behavior in terms of relativistic Lorentz contraction. The SG equation [Eq. (1)] obeys Einstein's theory of special relativity. Therefore, the lengths of moving objects should be contracted depending on their speeds. The center-of-mass speed of a moving breather alters the speeds of the kink and the antikink oscillating internally in the breather. This means that the widths of the kink and the antikink become different and the shape of the breather is distorted especially when the amplitude of the breather oscillation is small.

Figure 3 shows profiles of the phase field at different times when the amplitude of the breather is small. We can see that the breather shape is distorted and that it has peaks (red circles) and dips (blue triangles) simultaneously as shown in Fig. 3. When the amplitude relationship between the peak and dip changes, the position of the breather jumps from peak to dip and vice versa. This is the origin of the discontinuous points in the time evolution of R. In other words, our collective coordinate approach can partly incorporate relativistic effects even though it has nonrelativistic form.

V. EFFECTIVE INTERACTION

The Lagrangian of r and R [Eq. (25)] is still too complicated to allow us to grasp the general property of a breather pair. Numerical simulations show that the time evolution of r is not greatly altered by the breather interaction from that of the exact breather solution, i.e., Matsuda's identity [Eq. (6)]. Therefore, assuming Matsuda's identity for r, we average it out in time to derive a simple coarse-grained equation of motion for R, as explained in Appendix B.

The second and the fourth terms in Eq. (25) can be averaged analytically and give the effective potential for R. The third term becomes zero as a result of the time averaging due to the symmetry. Although we can also average r out in the first term and obtain an effective kinetic energy for R, the solution of the Euler-Lagrange equation of R obtained from the effective Lagrangian does not seem to be quantitatively correct. Therefore, we assume that the relevant part of the inertial mass of R comes simply from the reduced mass of the two breathers in the same way as the discussion of Eq. (22), namely, 8Γ.

Finally, we obtain the effective Lagrangian,

$$L_{\text{eff}}^{\pm} = 4\Gamma\dot{R}^2 - V_{\text{eff}}^{\pm}, \tag{33}$$

$$V_{\text{eff}}^{\pm} = \mp 64\frac{\Gamma^3}{\omega^2}\exp(-\Gamma|R|), \tag{34}$$

where the upper (lower) sign corresponds to the in-phase (out-of-phase) oscillation mode and $V_{\text{eff}}^{\pm}$ denotes the effective interaction between two breathers. Equation (34) clearly shows that the interaction between two breathers is attractive for the in-phase mode and repulsive for the out-of-phase mode. The major part of Eq. (34) comes from the integration of the product of two kink solutions, $\sigma[\Gamma\{x+(R-r)/2\}]\sigma[\Gamma\{x-(R-r)/2\}]$, for the out-of-phase mode and the product of the kink and antikink solutions, $\bar{\sigma}[\Gamma\{x+(R-r)/2\}]\sigma[\Gamma\{x-(R-r)/2\}]$, for the in-phase mode. This means that the sign and the strength of the breather-breather interaction are primarily determined by the interaction between the innermost two kinks (kink and antikink) in the case of the out-of-phase (in-phase) oscillation mode.

The Euler-Lagrange equation derived from $L_{\text{eff}}^{\pm}$ can be solved analytically. If the initial value of R is R_0, the solutions are given by

$$|R| = R_0 + \frac{2}{\Gamma}\ln[\cosh(\Omega_0 t)] \tag{35}$$

for the out-of-phase oscillation and

$$|R| = R_0 + \frac{2}{\Gamma}\ln\{\cos[\Omega_0(t - nT/2)]\},$$

$$(2n-1)T/4 \le t \le (2n+1)T/4, \quad n = 1,2,3,\ldots \tag{36}$$

for the in-phase oscillation, where

$$\Omega_0 = \frac{2\Gamma^2}{\omega}e^{-\Gamma R_0/2}, \tag{37}$$

$$T = \frac{2\pi}{\Omega}, \tag{38}$$

$$\Omega = \frac{\pi}{2\arccos(e^{-\Gamma R_0/2})}\Omega_0. \tag{39}$$

Figure 4 shows the time evolution of R obtained by numerical simulations of SG equation (solid red lines) together with analytic solutions [Eqs. (35) and (36)] (dotted blue lines) for various angular frequencies of the breather oscillation from ω=0.1 to ω=0.9 when the initial value of R is 15. The upper graphs (a) and (b) show the results for the out-of-phase modes and the lower graphs (c) and (d) show the results for the in-phase modes. Here, we only select data where the phase field amplitude is larger than 0.3. Solutions (35) and (36) are in good agreement with the results of the SG simulations over entire parameter ω range.

Solution (36) indicates that the in-phase oscillating breathers form a new kind of bound pair with the internal

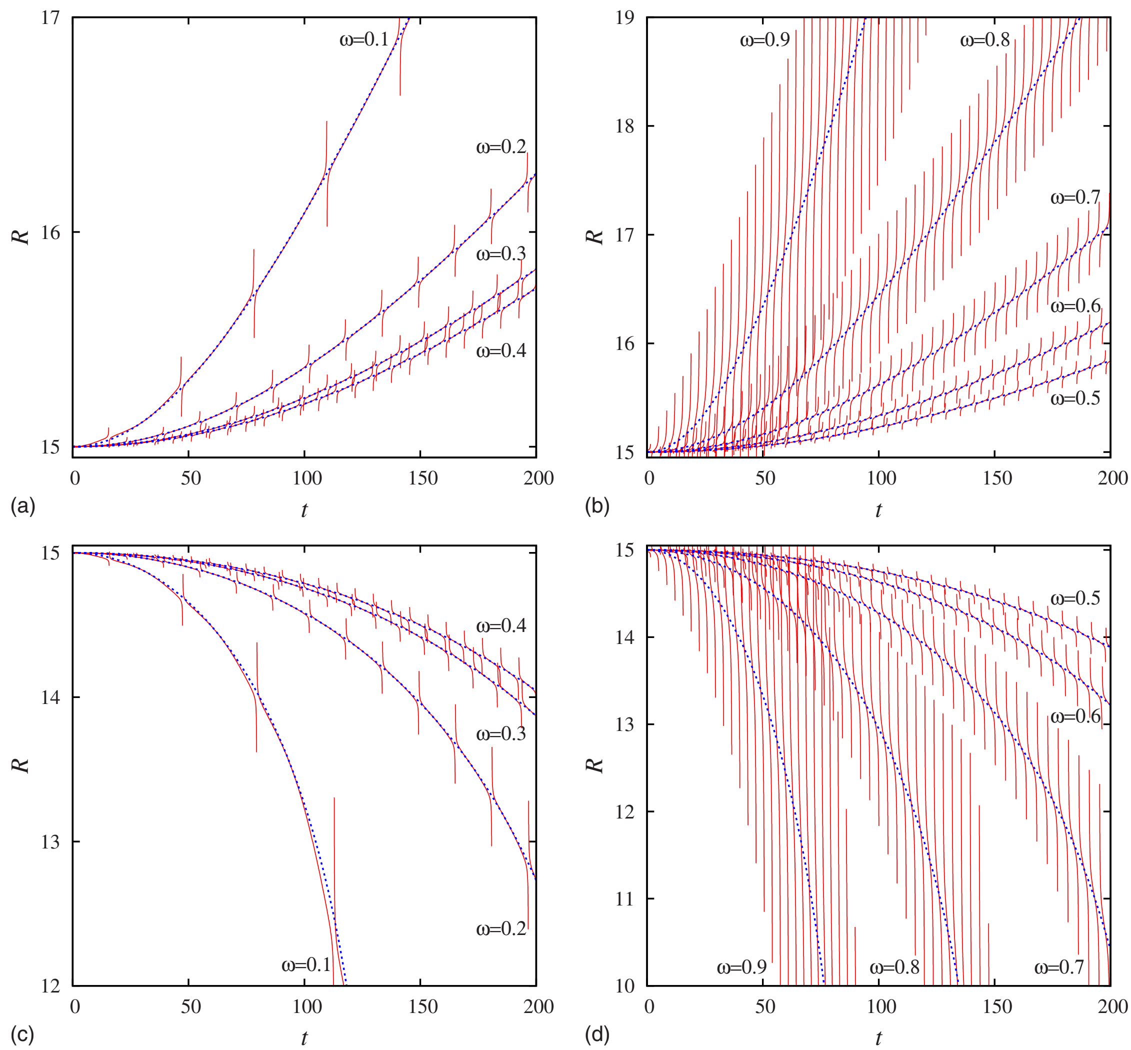

FIG. 4. (Color online) Time evolution of the separation between two breathers. The red lines show the results of the numerical simulation for the sine-Gordon equation. The dotted blue lines show the analytic solutions obtained by the collective coordinate method. Here, only data where the phase field amplitude is larger than 0.3 are selected.

oscillation mode whose angular frequency is given by Eq. (39). Figure 5 shows the results of an SG simulation for long-term time evolution (red circles) together with the results of Eq. (36) (blue lines). Here, we only select data where the phase field amplitude is larger than 0.8 so that the oscillating behavior can be clearly seen without being hidden by the jumping behavior caused by the relativistic effect. This figure clearly shows the existence of the expected internal oscillation mode of the in-phase breather pair.

Figure 6 shows the angular frequencies calculated from the results of SG simulations for various ω values and three different initial conditions, $R_0=10$, 12, and 15. Equation (39) accurately reproduces the numerical results.

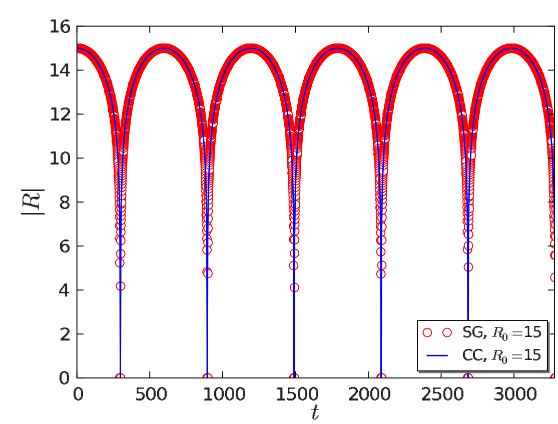

FIG. 5. (Color online) Long-term time evolution of the separation between the breathers oscillating in phase. The red circles show the results of numerical simulations of the sine-Gordon equation. The solid blue line shows the analytic solution obtained by the collective coordinate method. Here, only data where the phase field amplitude is larger than 0.8 are selected.

VI. DISCUSSION AND SUMMARY

Now, we discuss our results in comparison with those reported in [14,15]. Our effective interaction [Eq. (34)] has a similar property to the result in [14], namely, there is asymptotic exponential dependence on the breather separa-

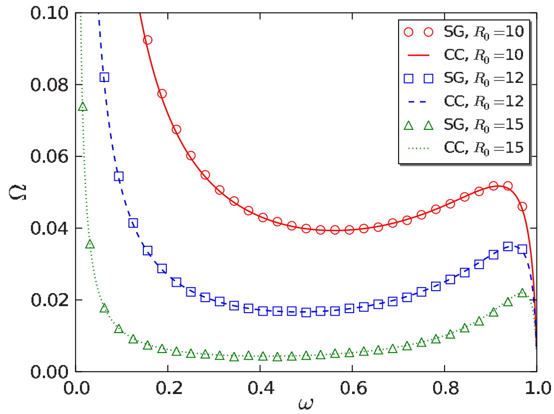

FIG. 6. (Color online) Angular frequencies of the internal oscillation of a bound breather pair for various ω values and three different initial conditions, $R_0=10$ (red circles, solid red line), 12 (blue squares, dashed blue line), and 15 (green triangles, dotted green line). The symbols show the results of numerical simulations of the sine-Gordon equation and the lines show the analytic solutions obtained by the collective coordinate method.

tion and power-law dependence on the breather frequency. However, the previous results reveal unexpected behavior in the breather-breather interaction V, which is defined by $V=E-E_{\mathrm{SB}}$, where

$$E=\left[16\Gamma-8m^2\frac{\Gamma^3}{\omega^2}+6m^4\frac{\Gamma^5}{\omega^4}\right]E(1-m^2), \tag{40}$$

$$E_{\mathrm{SB}}=16\Gamma, \tag{41}$$

with $E(1-m^2)$ being the complete elliptic integral of the second kind. The interaction for the in-phase oscillating pair changes from attractive to repulsive at the maximum point, as shown by the red circles in Fig. 7.

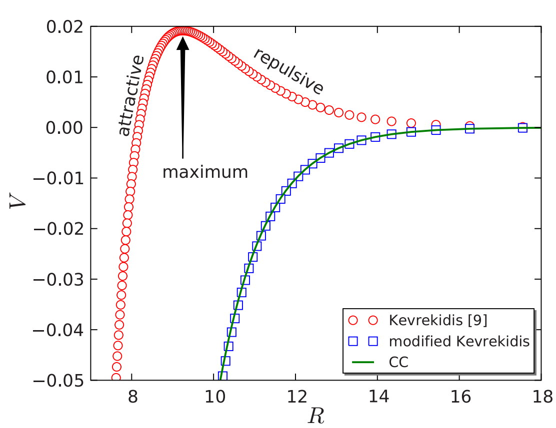

FIG. 7. (Color online) Asymptotic behavior of the interaction potential for the large separation between the breathers. The red circles show results obtained using the definition of the interaction potential in [14]. The blue squares show results obtained using the definition modified with Eq. (42). The green line shows the analytic solution obtained by the collective coordinate method.

There is an ambiguity in the determination of the asymptotic form of the interaction in the treatment described in [14]. The interaction between two breathers is determined by subtracting the energy of a single breather [Eq. (41)] from the energy of the breather lattice [Eq. (40)]. However, a single breather may possess additional energy that depends on the period of the breather lattice. It is possible that the additional energy produces no net force between two adjacent breathers, as noted at the end of Appendix B. Indeed, if we replace E_{SB} with

$$\bar{E}_{\mathrm{SB}}=16\Gamma E(1-m^2), \tag{42}$$

in Eq. (40), the result of our collective coordinate approach [Eq. (34)] (solid green line) is reproduced as shown by the blue squares in Fig. 7.

The equation of motion for R derived from our effective Lagrangian equation [Eq. (33)] is identical to Eq. (19) in [15] except that the time dependence of the effective interaction is averaged out in our equation. In Manton's approach used in [15], the force exerted on one breather by its neighbor is deduced from the time derivative of the momentum for an interval containing the breather. Therefore, this approach is justified only when the total momentum is a conserved quantity. Although Manton's approach is much simpler than ours for obtaining the asymptotic interaction in a translationally invariant system, our approach has greater applicability since it can also be used in inhomogeneous systems.

In summary, we have investigated the interaction between two breathers oscillating in or out of phase in an SG system using a CC approach. We have derived a set of ordinary differential equations for the CCs: the width of the individual breathers r and the separation between the centers of mass of the two breathers R. While the equations have singularity owing to the vanishing of the effective mass at the moments when the breathers become flat, we can remove the singularity by an appropriate coordinate transformation around the singular points and obtain the time dependence of R, which agrees reasonably well with the results obtained by the direct numerical simulation of the SG equation. Assuming that the time dependence of r is not affected by the breather interaction, we have derived a simple coarse-grained effective Lagrangian for R. The effective Lagrangian for R is simple enough to be solved analytically and correct enough to be used for a quantitative prediction of the long-term behavior. The analytic solution of the Euler-Lagrange equation of this effective Lagrangian provided us with an internal oscillation mode of the in-phase oscillating breather pair and we obtained an analytic expression for the angular frequency of this mode. This mode might be confirmed in spectroscopic experiments such as those found in [4,6,7].

ACKNOWLEDGMENTS

This work was supported in part by KAKENHI (Grant No. 18540352, No. 20540357, and No. 195836) from MEXT of Japan.

APPENDIX A: DERIVATION OF LAGRANGIAN FOR COLLECTIVE COORDINATES

In this section, we outline the derivation of the Lagrangian for the CCs R and r without the approximation, $\exp(-\Gamma|R|)\ll 1$.

The kinetic energy for R and r is derived by substituting Eq. (24) into Eq. (8). The time derivative of Eq. (24) reads

$$\frac{\partial}{\partial t}\phi_{\rm bp}^{\pm}(x,t) = \dot{X}_1\sigma'(y_1) - \dot{X}_2\sigma'(y_2) \mp \dot{X}_2\sigma'(y_3) \pm \dot{X}_1\sigma'(y_4), \tag{A1}$$

where the prime denotes the differentiation with respect to the arguments of the function, namely,

$$y_1 = \Gamma x + X_1,\quad y_2 = \Gamma x + X_2,\quad y_3 = \Gamma x - X_1,$$

$$y_4 = \Gamma x - X_2, \tag{A2}$$

with $X_1=\Gamma(R+r)/2$, $X_2=\Gamma(R-r)/2$, and thus

$$\sigma'(y) = \frac{2}{\cosh y}. \tag{A3}$$

Following the procedure described in [12], the integration in Eq. (8) can be carried out by using Eq. (10) and we obtain

$$K[\phi_{\rm bp}^{\pm}] = M_R^{\pm}\dot{R}^2 + M_r^{\pm}\dot{r}^2 + M_{Rr}^{\pm}\dot{R}\dot{r}, \tag{A4}$$

$$M_R^{\pm} = 8\Gamma[1 - f(\Gamma r)] \pm 4\Gamma\{f[\Gamma(R+r)] + f[\Gamma(R-r)] - 2f(\Gamma R)\}, \tag{A5}$$

$$M_r^{\pm} = 8\Gamma[1 + f(\Gamma r)] \pm 4\Gamma\{f[\Gamma(R+r)] + f[\Gamma(R-r)] + 2f(\Gamma R)\}, \tag{A6}$$

$$M_{Rr}^{\pm} = \pm 4\Gamma\{f[\Gamma(R+r)] - f[\Gamma(R-r)]\}. \tag{A7}$$

In the same way, the potential energy for R and r is derived by substituting Eq. (24) into Eq. (9). By using the relations [12],

$$\sin\sigma(y) = \sigma''(y) = -\frac{2\sinh y}{\cosh^2 y}, \tag{A8}$$

$$\cos\sigma(y) = \frac{\sigma'''(y)}{\sigma'(y)} = 1 - \frac{2}{\cosh^2 y}, \tag{A9}$$

and the formulas [Eqs. (10) and (11)], we obtain

$$\begin{aligned} V^{\pm}(R,r) = {}& 8(\Gamma+\Gamma^{-1})\{2 - 2f(\Gamma r) \pm 2f(\Gamma R) \mp f[\Gamma(R+r)] \mp f[\Gamma(R-r)]\} - 16\Gamma^{-1}[1-\cosh(\Gamma r)][1\pm\cosh(\Gamma R)] \\ & \times\Bigg[2\frac{\cosh(\Gamma r)\mp\cosh(\Gamma R)}{\cosh(\Gamma r)\pm\cosh(\Gamma R)}\left\{\frac{g[\Gamma(r)]}{1\mp\cosh(\Gamma R)} - \frac{g[\Gamma(R)]}{1+\cosh(\Gamma r)}\right\} \\ & + \frac{\{1\mp\cosh[\Gamma(R+r)]\}g[\Gamma(R+r)] + \{1\mp\cosh[\Gamma(R-r)]\}g[\Gamma(R-r)]}{[1+\cosh(\Gamma r)][1\mp\cosh(\Gamma R)]}\Bigg]. \end{aligned} \tag{A10}$$

APPENDIX B: TIME AVERAGING

Introducing new variables $y=\Gamma r$, $\theta=\omega t$, Matsuda's identity [Eq. (6)] reads

$$y(\theta) = 2\,\mathrm{arcsinh}\left[\frac{\Gamma}{\omega}\cos\theta\right] \tag{B1}$$

and the time average of a function of y, e.g., $F(y)$, is defined by

$$\langle F(y)\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} F[y(\theta)]d\theta. \tag{B2}$$

Using relations

$$\dot{y}^2 = \frac{4\Gamma^2\sin^2\theta}{1+\dfrac{\Gamma^2}{\omega^2}\cos^2\theta} = 4\left(\frac{1}{\cosh^2(y/2)} - \omega^2\right),$$

$$\cosh y = 2\cosh^2(y/2) - 1,\quad \sinh y = 2\sinh(y/2)\cosh(y/2),$$

the time average of the Lagrangian [Eq. (25)] can be written by using the following quantities:

$$\langle\cosh^2(y/2)\rangle = \frac{1+\omega^2}{2\omega^2}, \tag{B3}$$

$$\langle\cosh^{-2}(y/2)\rangle = \omega, \tag{B4}$$

$$\langle y\sinh y\rangle = \frac{\Gamma^2}{\omega^2} - \frac{2}{\omega^2}\ln\omega, \tag{B5}$$

$$\langle y\tanh(y/2)\rangle = -2\ln\omega, \tag{B6}$$

$$\left\langle y\frac{\sinh(y/2)}{\cosh^3(y/2)}\right\rangle = 2\omega(1-\omega), \tag{B7}$$

$$\langle f(y)\rangle = \frac{\omega}{\Gamma}\arcsin\Gamma. \tag{B8}$$

There are three remarks to be made as regard to the time-averaging procedure. First, the time average of $M_R^{\pm}$ is estimated as

$$\langle M_R^{\pm}\rangle = 8\Gamma - 8\omega \arcsin \Gamma \pm 16\Gamma e^{-\Gamma|R|}\left\{(\Gamma R-1)\frac{\Gamma^2}{\omega^2} + \frac{2}{\omega^2}\ln \omega\right\}. \tag{B9}$$

This estimate underestimates the effective inertial mass for R when $\Gamma R \ll 1$ since the R acceleration obtained from this estimation is larger than that obtained from the numerical simulation of the SG equation. Therefore, we adopt the reduced mass of the two breathers, 8Γ, as the effective inertial mass for R as noted in Sec. V. Some part of Eq. (25) would be irrelevant such as $L_{\text{kp}}^{(2)}$ in the Lagrangian for the kink-antikink pair [Eq. (17)].

Second, the time average of the third term of Eq. (25) is zero since $y(\pi)=y(-\pi)=-y(0)$ and $M_{Rr}^{\pm}$ are odd functions with respect to y,

$$\begin{aligned}\langle M_{Rr}^{\pm}(y,R)\dot{y}\rangle \dot{R}/\Gamma &= \dot{R}\frac{\omega}{2\pi\Gamma}\left[\int_{y(-\pi)}^{y(0)} M_{Rr}^{\pm}(y,R)dy + \int_{y(0)}^{y(\pi)} M_{Rr}^{\pm}(y,R)dy\right]\\ &= \dot{R}\frac{\omega}{\pi\Gamma}\left[\int_{-y(0)}^{y(0)}\{M_{Rr}^{\pm}(y,R) + M_{Rr}^{\pm}(-y,R)\}dy\right] = 0.\end{aligned} \tag{B10}$$

Finally, there are some cancellations between the second and the fourth terms of Eq. (25), which give the effective potential for R. This means that the R dependence of the kinetic energy for r does not necessarily produce potential energy for R. In other words, some part of the total energy is irrelevant to the interaction between the two breathers. The irrelevant part, such as Eq. (42), should be subtracted when the breather-breather interaction potential is estimated. This is performed automatically during the time averaging.

[1] E. Trías, J. J. Mazo, and T. P. Orlando, Phys. Rev. Lett. **84**, 741 (2000).
[2] P. Binder, D. Abraimov, A. V. Ustinov, S. Flach, and Y. Zolotaryuk, Phys. Rev. Lett. **84**, 745 (2000).
[3] H. Kleinert and K. Maki, Phys. Rev. B **19**, 6238 (1979).
[4] F. Fillaux, C. J. Carlile, and G. J. Kearley, Phys. Rev. B **58**, 11416 (1998).
[5] S. Adachi, V. M. Kobryanskii, and T. Kobayashi, Phys. Rev. Lett. **89**, 027401 (2002).
[6] R. Morisaki, T. Ono, H. Tanaka, and H. Nojiri, J. Phys. Soc. Jpn. **76**, 063706 (2007).
[7] E. Orignac, R. Citro, S. Capponi, and D. Poilblanc, Phys. Rev. B **76**, 144422 (2007).
[8] A. Sánchez and A. R. Bishop, SIAM Rev. **40**, 579 (1998).
[9] T. Matsuda, Lett. Nuovo Cimento Soc. Ital. Fis. **24**, 207 (1979).
[10] K. Forinash and C. R. Willis, Physica D **149**, 95 (2001).
[11] M. Taki, J. C. Fernandez, and G. Reinisch, Phys. Rev. A **38**, 3086 (1988).
[12] C. D. Ferguson and C. R. Willis, Physica D **119**, 283 (1998).
[13] T. Fujii, M. Nishida, and N. Hatakenaka, Phys. Rev. B **77**, 024505 (2008).
[14] P. G. Kevrekidis, A. Saxena, and A. R. Bishop, Phys. Rev. E **64**, 026613 (2001).
[15] P. G. Kevrekidis, A. Khare, and A. Saxena, Phys. Rev. E **70**, 057603 (2004).
[16] N. S. Manton, Nucl. Phys. B **150**, 397 (1979).
[17] M. B. Fogel, S. E. Trullinger, A. R. Bishop, and J. A. Krumhansl, Phys. Rev. Lett. **36**, 1411 (1976).
[18] J. G. Caputo and N. Flytzanis, Phys. Rev. A **44**, 6219 (1991).
[19] J. G. Caputo, N. Flytzanis, and C. N. Ragiadakos, J. Phys. Soc. Jpn. **63**, 2523 (1994).
[20] M. J. Ablowitz, B. M. Herbst, and C. M. Schober, Physica D **87**, 37 (1995).